THE DISTRIBUTION OF CURRENT IN A NETWORK OF CONDUCTORS (INTRODUCTION TO COMBINATORIAL TOPOLOGY)

HERMANN WEYL

The study of the continuum, Analysis situs[1], has a purely combinatorial portion which now, thanks to the fundamental work of H. Poincaré[2], can be studied independently and admits a complete, systematic exposition. I gave a series of lectures on this topic in 1918 at the Superior Technical School of Zurich. Since then, other works on the topic have been published, one in the same vein by O. Veblen[3], and another limited to the two-dimensional case by Chuard[4]. The (one-dimensional) problem of the distribution of current in an arbitrary network of conductors is quite appropriate as an introduction to the topic, since it highlights the fundamental concepts which then can be extended to the higher-dimensional case.

We will suppose that the electrical network is composed of a finite number of homogeneous wires which meet in a finite number of nodes. The related geometric construction will be called a *complex of segments*, the nodes will be the *vertices* of the complex, and the wires will be the edges of the complex.[5]

More rigorously: *A complex consists of a finite number of "vertices" or elements of dimension 0 and a number of "edges" or elements of dimension one. Each edge is bounded by two of these vertices and this boundary data completely describes the structure of the complex.*

Instead of saying "the vertex a bounds the edge σ," we will also say "σ ends at a" or "σ and a are incident elements of the complex." It is not necessary that three or more edges terminate in a given vertex a; it is possible that only two or even only one edge ends at a; it may also happen that a is an isolated vertex that does not bound any edge.

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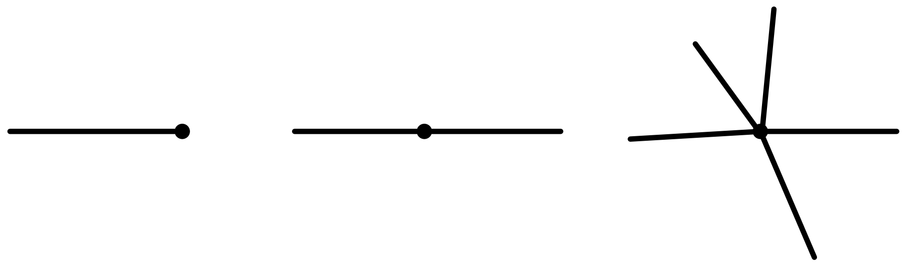
[1]Just around the time this paper was published, the term "topology" was replacing the now quaint term "analysis situs."

[2]Poincaré's seminal paper *Analysis situs* was published in 1895, and followed by a sequence of five supplements.

[3]Oswald Veblen gave a set of lectures in the Cambridge Colloquium, which were published in 1916.

[4]Jules Chuard, "Questions d'analysis situs," *Rendiconti del Circolo Matematico di Palermo* 46 (1922).

[5]I have translated the terms "punto" and "segmento" as "vertex" and "edge," respectively, since these sound more natural from the modern point of view. We expect a line segment to have an infinite number of points, after all.



We will also allow the complex to have no edges at all, that is, that it may be formed entirely of vertices, but we will exclude the "empty set" which contains neither vertices nor edges. In diagrams of complexes, which are used in combinatorial analysis situs (regardless of the actual nature of the elements forming the complex), elements must be distinguished from each other by some symbol, for example, in the complex formed from the edges and vertices of a tetrahedron (a Wheatstone bridge configuration) there are four vertices $0, 1, 2, 3$, and six edges $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, and the arrangement is

$$\alpha \begin{cases} 0 \\ 1 \end{cases}, \beta \begin{cases} 0 \\ 2 \end{cases}, \gamma \begin{cases} 0 \\ 3 \end{cases} \qquad \alpha' \begin{cases} 2 \\ 3 \end{cases}, \beta' \begin{cases} 3 \\ 1 \end{cases}, \gamma' \begin{cases} 1 \\ 2 \end{cases}$$

where, for instance, $\alpha \begin{cases} 0 \\ 1 \end{cases}$ should be read "α is bounded by 0 and 1."

A complex C may either be connected or consist of several pieces which are not connected to each other. A subset C' of the elements of C is called isolated when there is no pair of incident elements in C where one is in C' and the other is not. A complex is connected when its elements cannot be partitioned into two isolated subsets C' and C''.

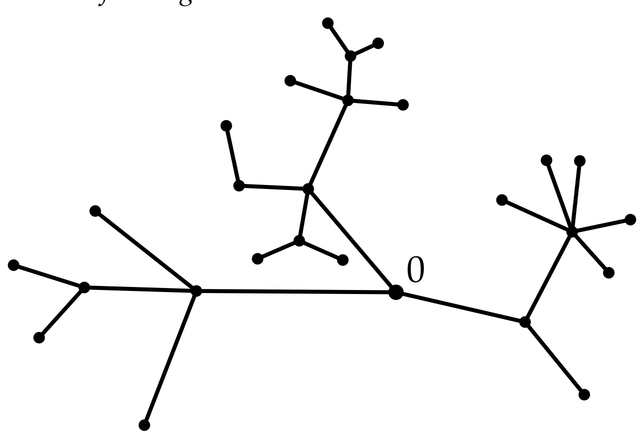
The edge σ bounded by two vertices a and b may be traversed in the two different directions from a to b or from b to a. A *chain* is a succession of directed edges, in which the endpoint of one edge is the initial point of the following edge. If we take the vertices through which the chain passes as well, we can define the chain in terms of an alternating series of elements of the complex: "vertex, edge, vertex, edge, . . . , vertex," in which every edge is neighbored by the two vertices in its boundary. A chain *joins* the first vertex of this sequence with the last, and it is *closed* if the first and the last vertex are the same; in this case the sequence is no longer considered as a linear order, but rather ordered cyclically, and the traversal can be considered as starting at any vertex. In general we do not assume that all elements of the sequence defining a chain are distinct; the chain may pass multiple times through the same vertex or edge. If all the elements of the series of elements are distinct, the corresponding chain is called *simple*, and a simple closed chain is called a *cycle* or circuit.

Every complex can be decomposed in precisely one way into a set of isolated, connected subcomplexes. Any such complex can be obtained as follows: a vertex 0 is selected, to which are added all edges leaving 0, followed by all vertices except 0 which are in the boundary of these edges, after which are added the edges which leave these vertices and have not yet been considered, and so on until no new elements are obtained. Thus we obtain the collection $C(0)$, corresponding to the element 0, which is obviously connected and isolated. It contains all vertices that can be joined to 0 by a chain (and only those vertices), and the construction proves that, if a chain goes from 0 to a, the points 0 and a can be joined by a simple chain. Not only may any point a of $C(0)$ be connected to a, but any two points a and a' of $C(0)$ may be connected, for which it is sufficient to connect a with 0 and 0 with a'.

As a result, if 1 is a point not contained in $C(0)$, the elements of the collection $C(1)$ are all distinct from the elements of $C(0)$. From this the theorem on the decomposition of a complex into connected, isolated subcomplexes results, and furthermore the following theorem has been demonstrated: *In a connected complex any two points may be linked by a simple chain.*[6]

It is well known that a stationary current cannot flow in a network of conductors if it does not contain a circuit, or a simple closed chain; but if there is one, an electromotive force applied to in the circuit suffices to produce a current. Thus, cycles play a decisive role in understanding the distribution of current. A connected complex with no cycles is called a *tree*.

If the collection $C(0)$ is constructed as previously described beginning at some point 0[7], in general several branches leave from any given vertex, but these never join together at a common endpoint. Each branch therefore gives a new vertex at its end; if we omit the *root vertex* 0 there are as many vertices as edges. That is, *the number of vertices in a tree is one greater than the number of its edges*.



A tree can also be characterized (among connected complexes) by the property that it breaks into separate parts when any segment is removed. If we remove a segment σ with endpoints a, b from a complex C, and the resulting complex C' is still connected, it is possible to join a and b with a simple chain in C', which together with σ forms a simple closed chain in C. Conversely, if C contains a cycle, C does not split in two when one edge

[6] It seems this argument doesn't quite prove that a simple chain suffices: if C is a tree and a, a' are on the same branch of the tree leaving from 0, the chain constructed is not simple. Weyl's theorem is still true as stated, but more justification is needed.

[7] For the next while, Weyl assumes that the complex C is a tree.

in the cycle is removed. From this we obtain a new proof of the theorem on the number of vertices and edges in a tree. Let N_0 be the number of vertices, N_1 the number of edges, and t the number of connected, isolated components of the complex. If the complex C has no cycles, the t parts of the complex are trees; on removing a segment the tree to which it belongs splits into two. Under this operation, which transforms the complex C into C', N_1 decreases by 1 and t increases by 1, and thus the number $N_1 + t$ is unchanged. C' is also a complex with no cycles. Now, removing the edges one by one until none remain, we reach a set C_0 consisting only of vertices, and therefore we have

$$N_1^0 = 0 \qquad N_0^0 = N_0 \qquad t^0 = N_0$$

As in every step $N_1 + t$ is invariant, we have

$$N_1 + t = N_1^0 + t^0 = N_0.$$

If the original complex was a tree, we have $t = 1$, and therefore

$$N_0 = N_1 + 1.$$

In order to approach the problem of the distribution of current we will suppose that each edge σ is provided with a direction of traversal. A current of intensity I^σ flows through each edge σ; this quantity is positive when the current flows in the positive direction, and negative otherwise. The edge σ conducts into a vertex a the quantity $\epsilon_{a\sigma} I^\sigma$ of current per unit of time, where $\epsilon_{a\sigma} = +1$ if a is the far end of σ when traversed in the positive direction, $= -1$ if it is at the near end, and $= 0$ if it is neither[8]. Kirchhoff's law, which states that the same amount of current leaves a as enters it, is as follows:

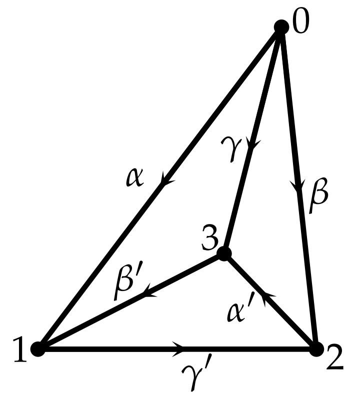
$$\sum_\sigma \epsilon_{a\sigma} I^\sigma = 0 \tag{1}$$

There are thus N_0 homogeneous linear equations in the N_1 unknowns I^σ. The matrix $E = |\epsilon_{a\sigma}|$ of their coefficients was first introduced by Poincaré in his paper "Analysis situs."

For the connections in the Wheatstone bridge, it is the following:

	α	β	γ	α'	β'	γ'
0	-1	-1	-1	0	0	0
1	$+1$	0	0	0	$+1$	-1
2	0	$+1$	0	-1	0	$+1$
3	0	0	$+1$	$+1$	-1	0

[8]That is, if a is not incident to σ.



What can be said about the linear independence of the equations (1)? Suppose that there is a linear identity with coefficients λ_a between the linear forms with the variables I^σ; that is, suppose

$$\sum_a \lambda_a \epsilon_{a\sigma} = 0$$

for every edge σ. These equations say that for the vertices a and b that bound an edge σ we have $\lambda_a = \lambda_b$, and from this we deduce as well that $\lambda_a = \lambda_b$ when a and b are two vertices that can be linked by a chain. If the complex is connected, the λ_a are all equal; therefore there is only one linear relation (with coefficients $\lambda_a = 1$) between the terms of (1), or in other words, the number of linearly independent equations is $N_0 - 1$. If the complex is a tree, this number is the same as N_1, the number of unknowns, and according to the theory of linear equations the only solution is $I^\sigma = 0$. This establishes the theorem: "A tree does not admit a stationary current."

If the electromotive forces are known, Ohm's law, which must hold for every closed chain in the network, will give us further linear equations to determine the current intensities. To determine the number of independent equations given by Ohm's law, we need to find the number of "independent" cycles in the network. This should be understood as follows: If, for example, a chain traverses an edge σ three times in the positive direction and five in the negative direction, we say that it traverses the edge a total of $3 - 5 = -2$ times. A chain associates to every edge σ an integer i^σ, its indicator, which counts how many times it is traversed in total by said chain. A chain is considered to be zero if all its indicators i^σ are zero; two chains are considered to be equivalent if their indicators i^σ are the same. This way of looking at things is suited to our purpose, since for a chain equivalent to zero Ohm's law gives the trivial formula $0 = 0$.

Two chains $\mathbf{i}_1$, $\mathbf{i}_2$ can be added together. If a is any vertex of the first chain and b any vertex in the second chain, it is sufficient to insert a connecting chain between a and b (requiring the complex in question to be connected). We first traverse $\mathbf{i}_1$, next the connecting chain from a to b, then $\mathbf{i}_2$, and finally return by the same chain from b to a; the closed chain thus obtained is the sum $\mathbf{i}_1 + \mathbf{i}_2$. The choice of the points a and b and of the connecting chain has no influence on the sum; with respect to the equivalence of chains it is uniquely determined. If i_1^σ are the indicators corresponding to the edges in $\mathbf{i}_1$ and i_2^σ those of $\mathbf{i}_2$, then $i_1^\sigma + i_2^\sigma$ are the indicators of $\mathbf{i} = \mathbf{i}_1 + \mathbf{i}_2$. A chain represented by the numbers i^σ is closed when, and only when, for each vertex as many edges of the chain enter as leave; that is, if for each vertex a the equation

$$\sum_\sigma \epsilon_{a\sigma} i^\sigma = 0 \tag{2}$$

holds.

The integer solutions of the equations (1) give us the closed chains in the network. The problem of the independence of closed chains is the same as the problem of the linearly independent solutions of these equations.

We now make a conceptual advance by defining a tree not as a connected complex with no closed simple chains, but as a complex in which every closed chain is equivalent to zero. To prove the equivalence of these definitions, it is necessary to show that: If a complex has no cycles, every closed chain is equivalent to zero. This may be deduced from the theory of linear equations; we have already seen that the equations (2) have the unique solution $i^\sigma = 0$ when the complex is a tree in the original sense. The same result may be obtained by a simple construction. The vertices and edges of a closed chain $\mathbf{i}$ form a cyclic sequence. If there are no simple closed chains, some element must appear multiple times in the sequence $\mathbf{i}$. If when traversing it beginning at some vertex the first element encountered twice were a vertex a, the part of the chain from a to a would be a simple cycle. Thus this element must be an edge σ, and $\mathbf{i}$ contains the sequence $\ldots a\,\sigma\, b\,\sigma\, a \ldots$ (the chain reverses at vertex b); since if there were other elements than b before σ was repeated

$$\ldots a\,\sigma\, b \ldots a'\,\sigma\, b' \ldots$$

then a' would have to coincide with a or b and the element a' would be repeated before σ. Therefore we can remove from the sequence the portion $\sigma\, b\,\sigma\, a$ and the given closed chain is reduced to an equivalent chain whose sequence of vertices and edges has been shortened by four elements. This operation may be repeated until the closed chain has been reduced to zero.

If we disassemble the complex by removing its edges one by one, in each step the number N_1 decreases by 1 and t increases by 0 or 1. If zero appears g times, the number $N_1 + t$ is decreased by 1 g times and remains the same $N_1 - g$ times; therefore

$$(N_1 + t) - g = N_0.$$

For a connected complex we have, in particular,

$$g = N_1 - N_0 + 1. \tag{3}$$

For a connected complex the number g defined by (3) is always ≥ 0, and and in whatever way the complex might be disassembled it always happens the same number of times that when an edge is removed there is no new component. In particular, the operation may be conducted in such a way that after the first g steps the complex remains connected and, therefore, after removing the first g edges it is reduced to a tree. As long as $g \neq 0$ the complex has not yet been converted into a tree and it is still possible to remove another edge without disconnecting it. For closed chains we obtain the following theorem: *"There are g simple closed chains $\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_g$ such that every closed chain is equivalent to one, and only one, linear combination*

$$m_1\mathbf{i}_1 + m_2\mathbf{i}_2 + \cdots + m_g\mathbf{i}_g \quad (m_1, \ldots, m_g \text{ integers})$$

of these chains."

Proof: Let $g > 0$. Then in the given connected complex C there exists a simple closed chain $\mathbf{i}_1$. Let $\sigma_1 = ab$ be an edge belonging to $\mathbf{i}_1$; when σ_1 is removed, C becomes a connected complex C' whose corresponding number is $g' = g - 1$. a and b are linked in C' by a simple chain $\mathbf{i}'_1$ which is obtained from $\mathbf{i}_1$ by removing σ_1. Any closed chain $\mathbf{v}$ in C passes a total of m_1 times through σ_1. We convert $\mathbf{v}$ into a chain $\mathbf{v}'$ of C' by, every time $\mathbf{v}$ passes through the edge σ_1 that connects a and b, replacing it with the path $-\mathbf{i}'_1$.[9] It is thus obvious that

$$\mathbf{v} = m_1\mathbf{i}_1 + \mathbf{v}' \quad (= \text{ representing equivalence}).$$

If $g - 1 > 0$ as well, it is possible to find a simple closed chain $\mathbf{i}_2$ in C' and convert C' into a connected complex C'' by removing an edge σ_2 from $\mathbf{i}_2$; thus we have

$$\mathbf{v} = m_1\mathbf{i}_1 + m_2\mathbf{i}_2 + \mathbf{v}'',$$

where $\mathbf{v}''$ is contained in C''. Proceeding in this fashion, we easily obtain g simple closed chains $\mathbf{i}_1, \ldots, \mathbf{i}_g$ and a connected complex $C^{(g)}$ such that every closed chain $\mathbf{v}$ in C may be represented in the form

$$\mathbf{v} = (m_1\mathbf{i}_1 + m_2\mathbf{i}_2 + \cdots + m_g\mathbf{i}_g) + \mathbf{v}^{(g)}$$

where $\mathbf{v}^{(g)}$ is contained in $C^{(g)}$. But $C^{(g)}$ is a tree, and so $\mathbf{v}^{(g)}$ is equivalent to zero.

The theory of linear equations shows that the equations (1) and (2) with integer coefficients have

$$g = N_1 - N_0 + 1$$

independent integer solutions

$$\mathbf{i}_1 = (i_1^\sigma) \quad \mathbf{i}_2 = (i_2^\sigma) \ldots \quad \mathbf{i}_g = (i_g^\sigma)$$

which linearly combine to form every solution, since N_1 is the number of unknowns and $N_0 - 1$ the number of independent equations. The theory of arithmetic completes this theorem thus (for any homogeneous system of linear equations with integer coefficients): the

[9] The negative sign comes from the implicit orientation that this path receives from $\mathbf{i}$.

basic integer solutions $\mathbf{i}_1, \dots, \mathbf{i}_g$ may be chosen in such a way that every integer solution $\mathbf{i}$ is a linear combination of these in the form

$$\mathbf{i} = m_1\mathbf{i}_1 + m_2\mathbf{i}_2 + \cdots + m_g\mathbf{i}_g$$

where the coefficients m are *integers*.

In our case, this means that every closed chain may be constructed from g independent chains $(\mathbf{i}_1 \dots \mathbf{i}_g)$. We have here constructed one such basis for these closed chains; our result has as an advantage over the result obtained from the general theory of linear equations the fact that the basis so constructed consists of simple closed chains. When we consider a greater number of dimensions it would be very difficult to follow these constructions and, therefore, we will prefer to base our results on the theory of linear equations. With the introduction of the matrix $E = \|\epsilon_{a\sigma}\|$ the most complicated combinatorial problems become problems accessible to Mathematics through the simple and well developed formalism of Algebra.

If E_σ is the electromotive force introduced across the edge σ, with r_σ the resistance of this wire, Ohm's law[10] applied to the circuit $\mathbf{i}_h$ gives

$$\sum_\sigma i_h^\sigma r_\sigma I^\sigma = \sum_\sigma i_h^\sigma E_\sigma \quad (h = 1, 2, \dots, g) \tag{4}$$

If we can show that the $N_0 - 1$ independent equations in the homogeneous system (1), together with the g inhomogeneous equations (4), form a system of $N_0 - 1 + g = N_1$ independent equations, it will follow that these uniquely determine the N_1 unknowns I^σ. And this may be deduced from the fact that *this problem is none other than orthogonal projection onto an N_1-dimensional space*.

A system of numbers $\mathbf{I} = (I^\sigma)$ associated with the edges σ of our network of conductors will be denoted by the name vector. In particular, the sought for current distribution is such a *vector*. The *scalar product* of two vectors $\mathbf{I} = (I^\sigma)$ and $\bar{\mathbf{I}} = (\bar{I}^\sigma)$ is the bilinear form

$$(\mathbf{I}\bar{\mathbf{I}}) = \sum_\sigma r_\sigma I^\sigma \bar{I}^\sigma.$$

When $(\mathbf{I}\bar{\mathbf{I}})$ vanishes, we say that the vectors $\mathbf{I}$ and $\bar{\mathbf{I}}$ are perpendicular. The corresponding quadratic form

$$(\mathbf{I}\mathbf{I}) = \sum_\sigma r_\sigma (I^\sigma)^2$$

is positive definite and represents the Joule effect per unit of time[11] generated by the current distribution (I^σ). The equations (1) define a g-dimensional variety of vectors[12] Γ in N_1-dimensional space, for which the g independent vectors

$$\mathbf{i}_1 = (i_1^\sigma), \quad \mathbf{i}_2 = (i_2^\sigma), \dots, \quad \mathbf{i}_g = (i_g^\sigma)$$

[10] In modern terms, we would probably say that this results from applying Ohm's law individually to each wire in the closed cycle $\mathbf{i}_h$.

[11] We would today just call this the power dissipated in the circuit; the Joule effect more specifically refers to the heat generated by this dissipation.

[12] This is a subspace in modern terminology.

form a basis.

The current distribution vector $\mathbf{I}$ belongs to this variety; that is, we have

$$\mathbf{I} = \lambda_1 \mathbf{i}_1 + \lambda_2 \mathbf{i}_2 + \cdots + \lambda_g \mathbf{i}_g \quad (I^\sigma = \lambda_1 i_1^\sigma + \lambda_2 i_2^\sigma + \cdots + \lambda_g i_g^\sigma) \tag{5}$$

If only the electromotive force E_σ were imposed, it would induce a current $I_0^\sigma = \frac{E_\sigma}{r_\sigma}$ in the wire σ. If we introduce the vector $\mathbf{I}_0 = (I_0^\sigma)$, the equations (4) say that the vector $\mathbf{I}_0 - \mathbf{I}$ is perpendicular to the variety Γ (i.e. to all the vectors in Γ). We therefore wish to decompose the given vector $\mathbf{I}_0$ into two components

$$\mathbf{I}_0 = \mathbf{I} + \mathbf{I}',$$

of which the first $\mathbf{I}$ belongs to the variety and the second $\mathbf{I}'$ is perpendicular to it; according to the theorems of Analytic Geometry, this problem of orthogonal projection always has a unique solution. It is obtained by substituting (5) into (4)[13]

$$\sum_{k=1}^{g} (\mathbf{i}_h, \mathbf{i}_k)\lambda_k = (\mathbf{i}_h, \mathbf{I}_0) \quad (h = 1, 2, \ldots, g) \tag{6}$$

We thus have g linear equations for the g unknowns λ with symmetric coefficients $(\mathbf{i}_h, \mathbf{i}_k)$; these always have one and only one solution, because the corresponding homogeneous equations

$$\sum_{k=1}^{g} (\mathbf{i}_h, \mathbf{i}_k)\lambda_k = 0 \tag{7}$$

only have the solution $\lambda = 0$. This is because, multiplying (7) by λ_h and summing with respect to h, we obtain for the vector $\mathbf{I}$ defined by the equation (5)

$$(\mathbf{I}\mathbf{I}) = 0.$$

Because the Joule effect is always positive, it follows from here that $\mathbf{I} = 0$ and therefore $\lambda_1 = \cdots = \lambda_g = 0$. The problem of the distribution of current arises in this way as one of the most beautiful applications of n-dimensional geometry.

Kirchhoff has given a different solution to the problem; his method calculates the determinant D of the equations (6). If $1, 2, \ldots, g$ is any set of g edges whose removal converts the network of conductors into a connected tree,

$$D = \sum (r_1 \cdot r_2 \cdot \cdots \cdot r_g)$$

where the sum is over all such sets of edges. From this it follows that $D \neq 0$ and is positive.

Translated by JAKOB HANSEN

[13] In modern notation, this forms the normal equations $A^* A \lambda = A^* \mathbf{I}_0$, where A is the matrix whose columns are the indicators for the basis cycles $\mathbf{i}_k$, and A^* its adjoint with respect to the inner product defined by the resistances.